# MATH 504 HOMEWORK 6

Due Wednesday, April 20.

**Problem 1.** *Suppose that for all* $n$, $2^{\aleph}_n = \aleph_{\omega+1}$. *Show that* $2^{\aleph_\omega} = \aleph_{\omega+1}$.

Hint: For every $A \subset \aleph_\omega$, define the $A_n = A \cap \aleph_n$. Consider the map $h(A) = \langle A_n \mid n < \omega \rangle$.

**Problem 2.** *Suppose that* $\kappa$ *is inaccessible in* $V$ *and* $\mathbb{P}$ *is a poset of cardinality less than* $\kappa$. *Let* $G$ *be* $\mathbb{P}$*-generic. Show that in* $V[G]$, $\kappa$ *remains inaccessible.*

**Problem 3.** *Let* $\kappa$ *be a regular uncountable cardinal and* $\mathbb{P}$ *be a* $\kappa$*-closed poset. Show that* $\mathbb{P}$ *preserves stationary subsets of* $\kappa$, *i.e. if* $S \subset \kappa$ *is stationary in the ground model, then* $S$ *remains stationary in any* $\mathbb{P}$*-generic extension.*

Hint: Given $S$, a name $\dot{C}$, and $p$, such that $p \Vdash$ "$\dot{C}$ is a club subset of $\kappa$", show there is a sequence in the ground model $\langle p_\alpha, \gamma_\alpha \mid \alpha < \kappa \rangle$, such that:

- $\langle p_\alpha \mid \alpha < \kappa \rangle$ is a decreasing sequence below $p$,
- $\langle \gamma_\alpha \mid \alpha < \kappa \rangle$ is a club in $\kappa$,
- each $p_\alpha \Vdash \gamma_\alpha \in \dot{C}$.

Then use stationarity of $S$ in the ground model.

**Problem 4.** *Let* $S \subset \omega_1$ *be a stationary set. Define* $\mathbb{P} := \{p \subset S \mid p$ *is closed and bounded*$\}$, *and set* $p \leq q$ *if* $p$ *end extends* $q$ *i.e. for some* $\alpha$, $p \cap \alpha = q$.

1. *Show that* $\mathbb{P}$ *is* $\omega$*-distributive, i.e. if* $p \Vdash \dot{f} : \omega \to ON$, *then there is some* $q \leq p$ *and a function* $g$ *in the ground model, such that* $q \Vdash \dot{f} = \check{g}$. *Note that this implies that* $\mathbb{P}$ *adds no countable subsets of* $\omega_1$, *and hence it preserves* $\omega_1$.
2. *What is the best chain condition for* $\mathbb{P}$*? Justify your answer. Use that and the above to show that* $\mathbb{P}$ *preserves all cardinals.*
3. *Suppose that* $T := S \setminus \omega_1$ *is also stationary. Let* $G$ *be a* $\mathbb{P}$*-generic filter. Show that in* $V[G]$, $T$ *is nonstationary.*

*Remark* 1. The above is an example of a forcing that destroys a stationary set, without collapsing cardinals. On the other hand you cannot use forcing to destroy a club set. More precisely, if $V \subset W$ are two models of set theory and $V \models$ "$D$ is club in $\kappa$", then $W \models$ "$D$ is club in $\kappa$". Note that in the above problem it was important that $S$ was stationary; i.e. you cannot add a new club through a nonstationary set.